Revisiting Fracture Mechanic Handbooks with (Interpretable) Sparse Regression

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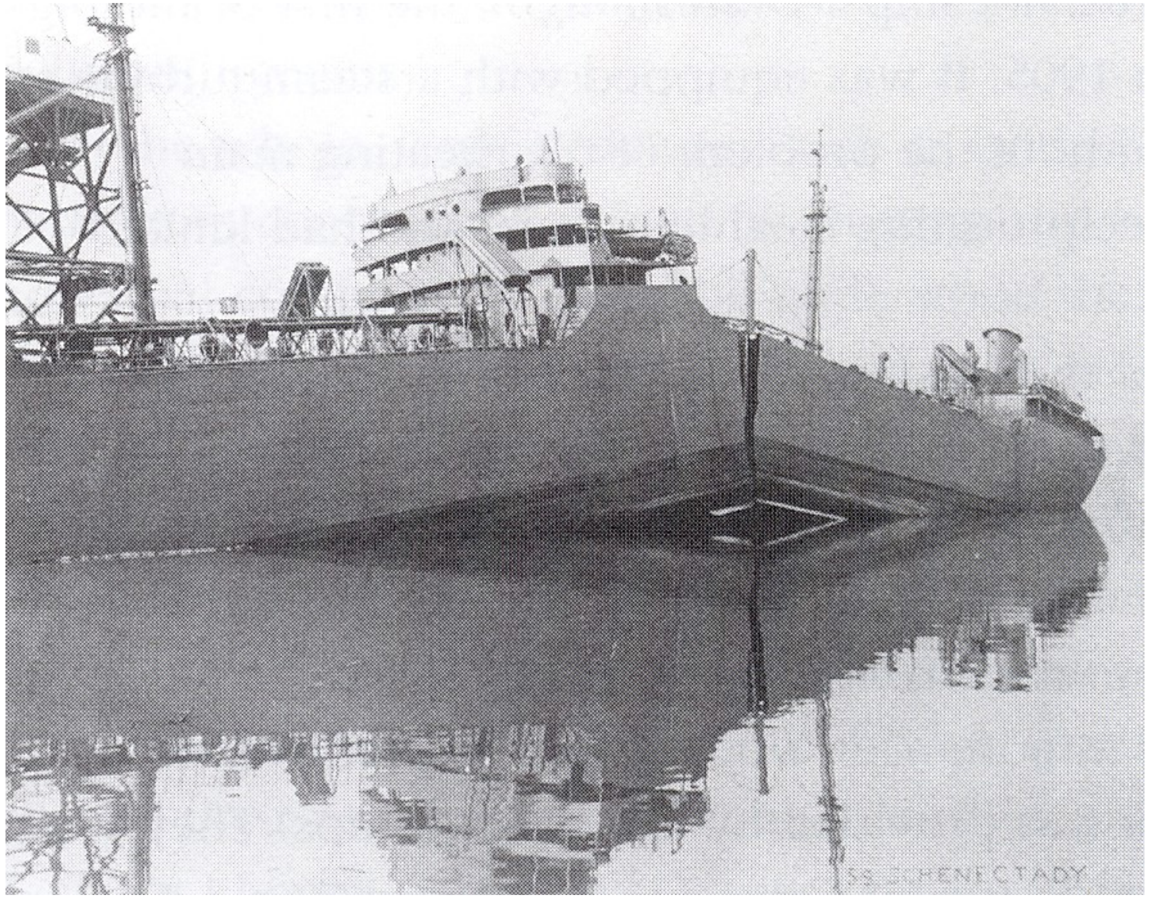
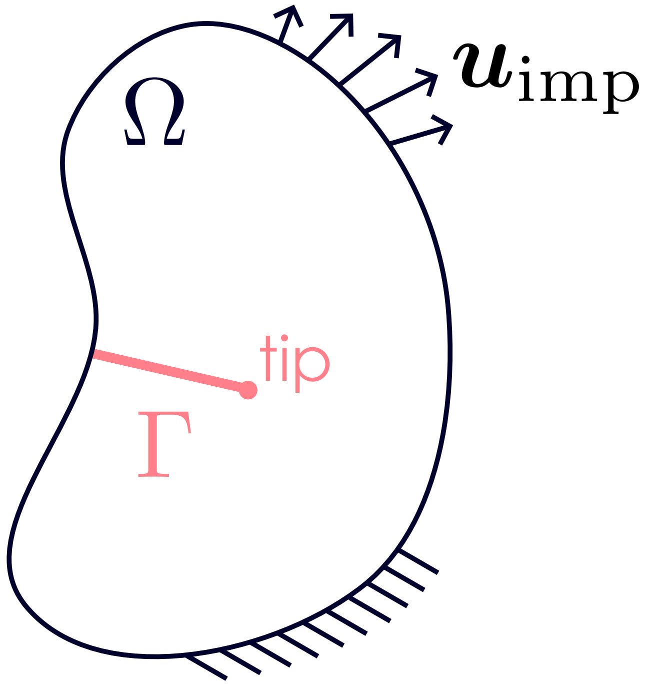
Linear Elastic Fracture Mechanics



The S.S. Schenectady tanker split apart by brittle fracture while in harbor, 1943.
Source : https://en.wikipedia.org/wiki/Fracture_mechanics.

TYPICAL PROBLEM

An elastic body Ω with a crack Γ under a given load.

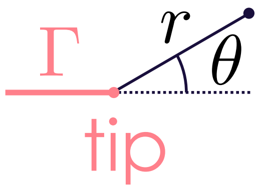


Assumption : Non-linearities are confined close to the crack tip.

QUESTION : Does the crack propagates ?

Crack tip singularity

Stress field (Williams, 1957)



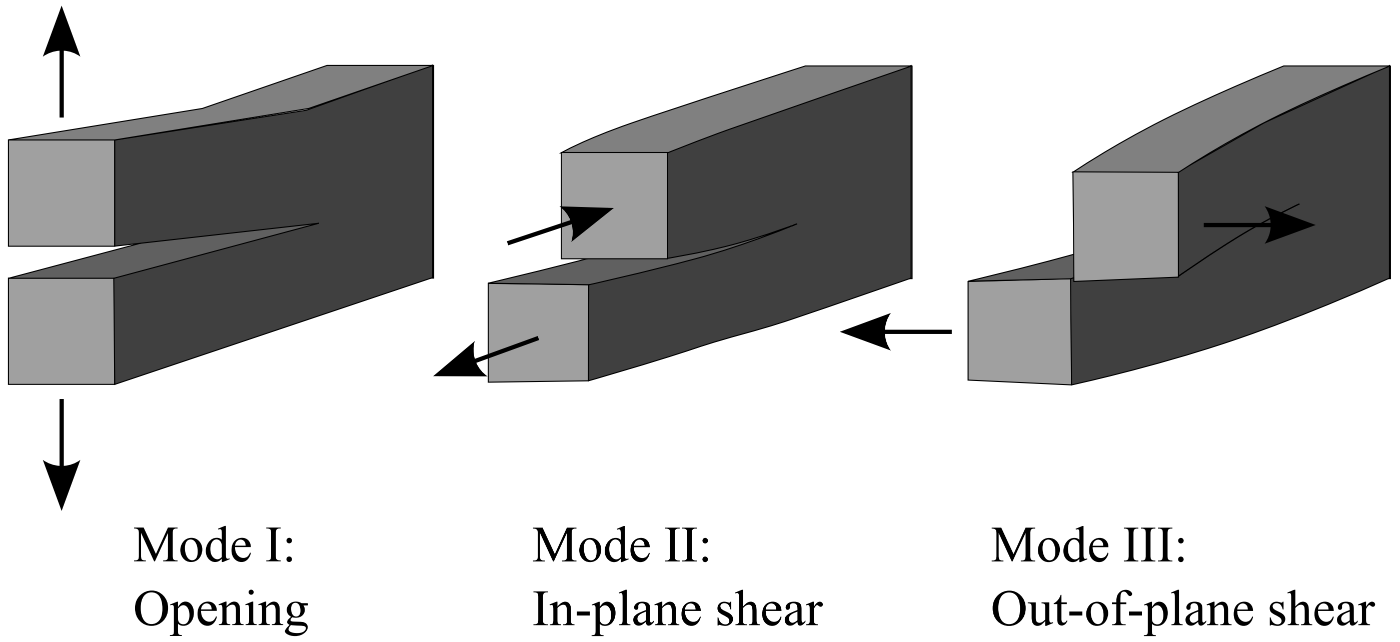
$$\sigma_{ij}(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} f_{ij}^{I}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f_{ij}^{II}(\theta) + \frac{K_{III}}{\sqrt{2\pi r}} f_{ij}^{III}(\theta) + \mathcal{O}(1), \quad \sigma_{ij}(r,\theta) \underset{r\to 0}{\to} \infty.$$

There is a stress singularity at crack tip.

Stress Intensity Factors (SIFs)

Denoted K_m, they :

- represent the crack tip sollicitation,
- depend on geometry and load.



Source: https://en.wikipedia.org/wiki/Fracture_mechanics.

Designing structures with cracks

Crack propagation criteria

Propagation criteria rely on the SIFs K_m, *e.g.*,

$K_I \leq K_{Ic}$, or

$$G = \frac{{K_I}^2 + {K_{II}}^2}{E'} + \frac{{K_{III}}^2}{2\mu} \leq G_c.$$

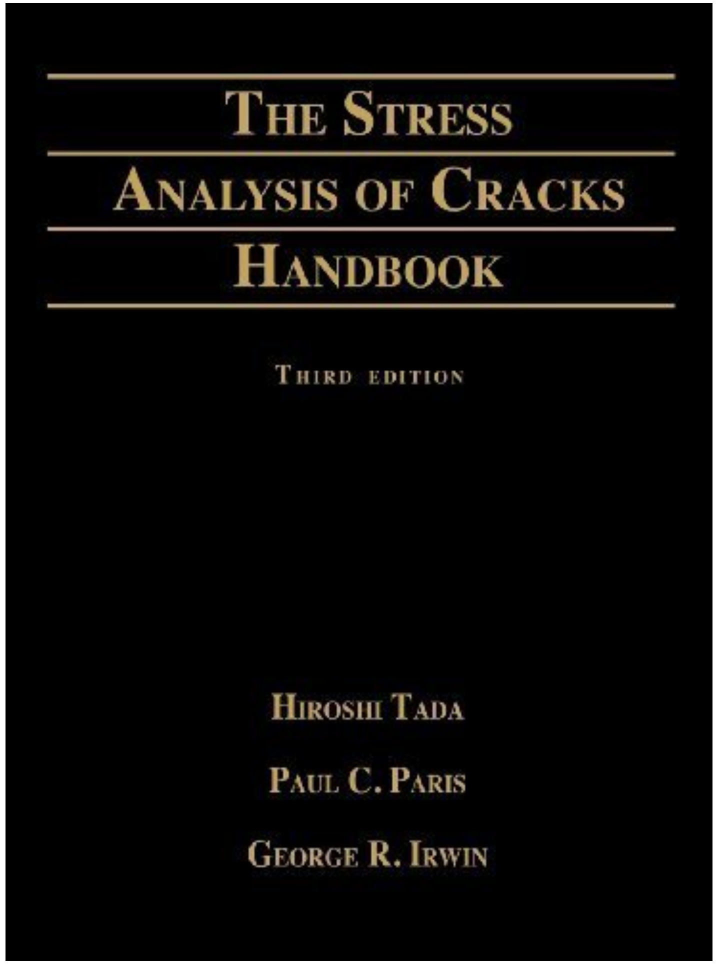
How to determine the SIFs ?

Different methods are employed :

- analytical (mostly for simple case, or (semi-)infinite domains),
- numerical (*e.g.*, FEM simulations),
- experimental (*e.g.*, photo-elasticity, DIC.).

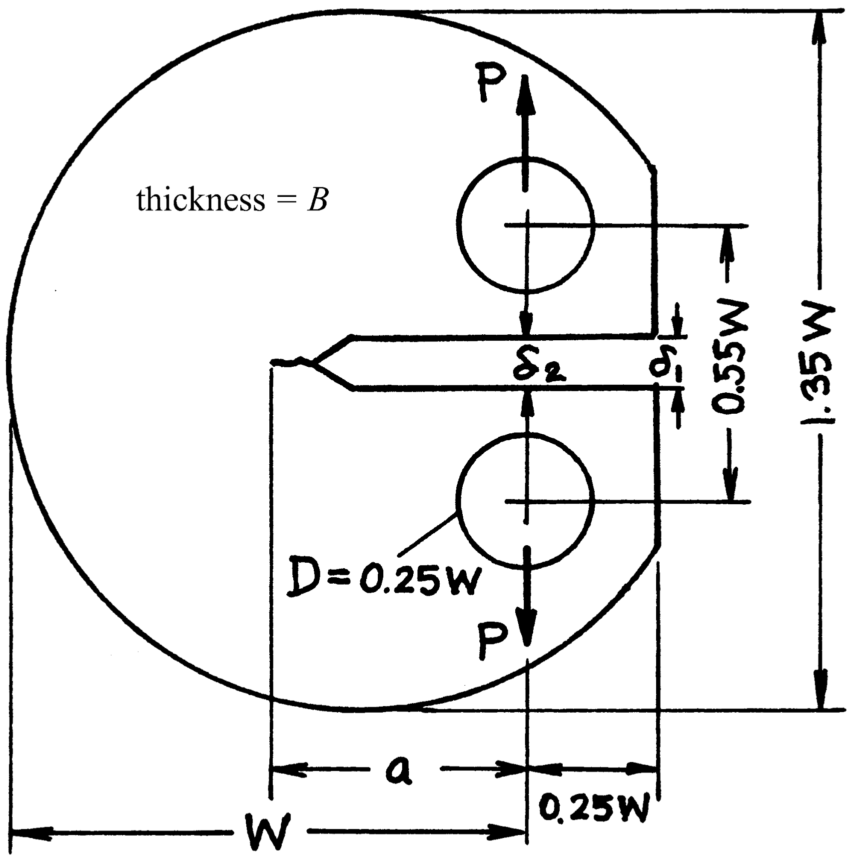
$\longrightarrow$ Known solutions have been compiled in handbooks!

Fracture mechanic design handbooks (Tada et al., 2000)



Example (Tada et al., 2000)

THE ROUND (DISK-SHAPED) COMPACT SPECIMEN



Method: Boundary Collocation Method
Accuracy: K_I 0.3% for $0.2 \leq A \leq 1.0$; δ_1 and δ_2 0.5% for $0.2 \leq A \leq 0.8$
References: **Newman 1979a, 1981b**

Methodology in handbook

1. Dimensional analysis

$$K_I = \bar{\sigma}\sqrt{W}\hat{K}_I(A), \quad \bar{\sigma} = \frac{P}{WB}, \quad A = \frac{a}{W}.$$

2. Data gathering (not if analytical method)
   - Observations : $(A, \hat{K}_I)$
   - Experimental/Numerical methods
3. Fit polynomial solution to data

$$\hat{K}_I(A) = \frac{(2 + A)(0.76 + 4.8A - 11.58A^4 + 11.43A^3 - 4.08A^4)}{(1 - A)^{3/2}}$$

Limitations

Dense polynomials, limited to few parameters, etc.

Finding a sparse analytical model

Colors: $\mathbf{Inputs\ (parameters)\ p}$, $\mathbf{Coefficients\ c}$, $\mathrm{Outputs}\ K_m$.

Objective

For a structure and load parameterized by $\mathbf{p} = [p_1, p_2, \dots, p_N]^T$, find the relationship $K_m(\mathbf{p})$.

Tool : LASSO regression (Tibshirani, 1996)

We look for a polynomial approximation of degree d (hyperparameter)

$$K_m(\mathbf{p}) \approx \widetilde{K}^{\mathbf{c}}_m(\mathbf{p}) = \sum_{i_1} \sum_{i_2} \cdots \sum_{i_N} c_{i_1, i_2, \dots, i_n} {p_1}^{i_1} {p_2}^{i_2} \dots {p_N}^{i_P}.$$

For N_{obs} observations ($\mathbf{p}$, K_m), the optimal coefficients $\mathbf{c}^*$ minimizes

$$\mathbf{c}^* = \arg\min_{\mathbf{c}} \frac{1}{2N_{\text{obs}}} \sum_{\boldsymbol{p}} \left(\widetilde{K}^{\mathbf{c}}_m(\mathbf{p}) - K_m(\mathbf{p}) \right)^2 + \beta \|\mathbf{c}\|_1, \quad \|\boldsymbol{c}\|_1 = \sum_n |c_n|,$$

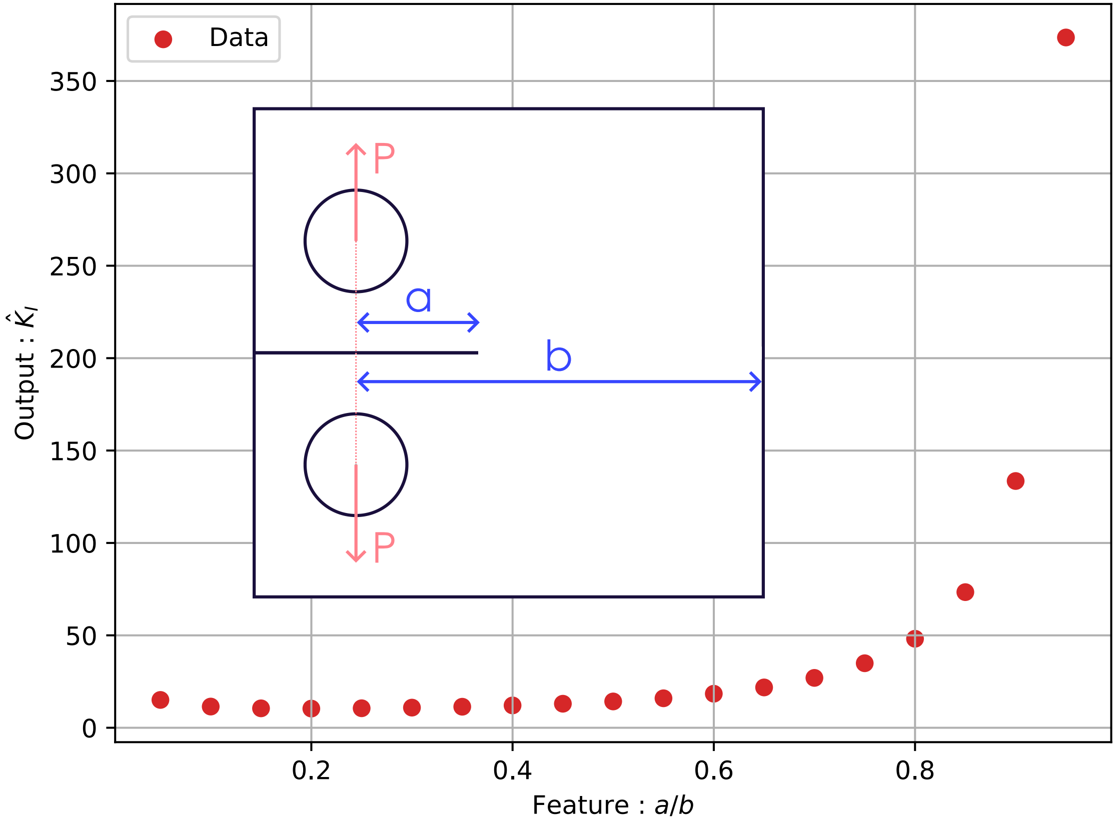
where β is a hyperparameter. The L1-norm of the coefficients $\mathbf{c}$ drives irrelevant coefficients to zero.

Adding non-polynomial terms to model

Replace $\mathbf{p}$ with $\mathbf{s} = [p_1, p_2, \dots, p_N, f(p_i)]$.

Ex. 1 : CT specimen - Setup & Data

Parameters and Observations



1. Dimensional analysis

$$K_I = \sigma\sqrt{a}\hat{K}_I\left(\frac{a}{b}\right), \quad \sigma = \frac{P}{b}.$$

2. Data gathering

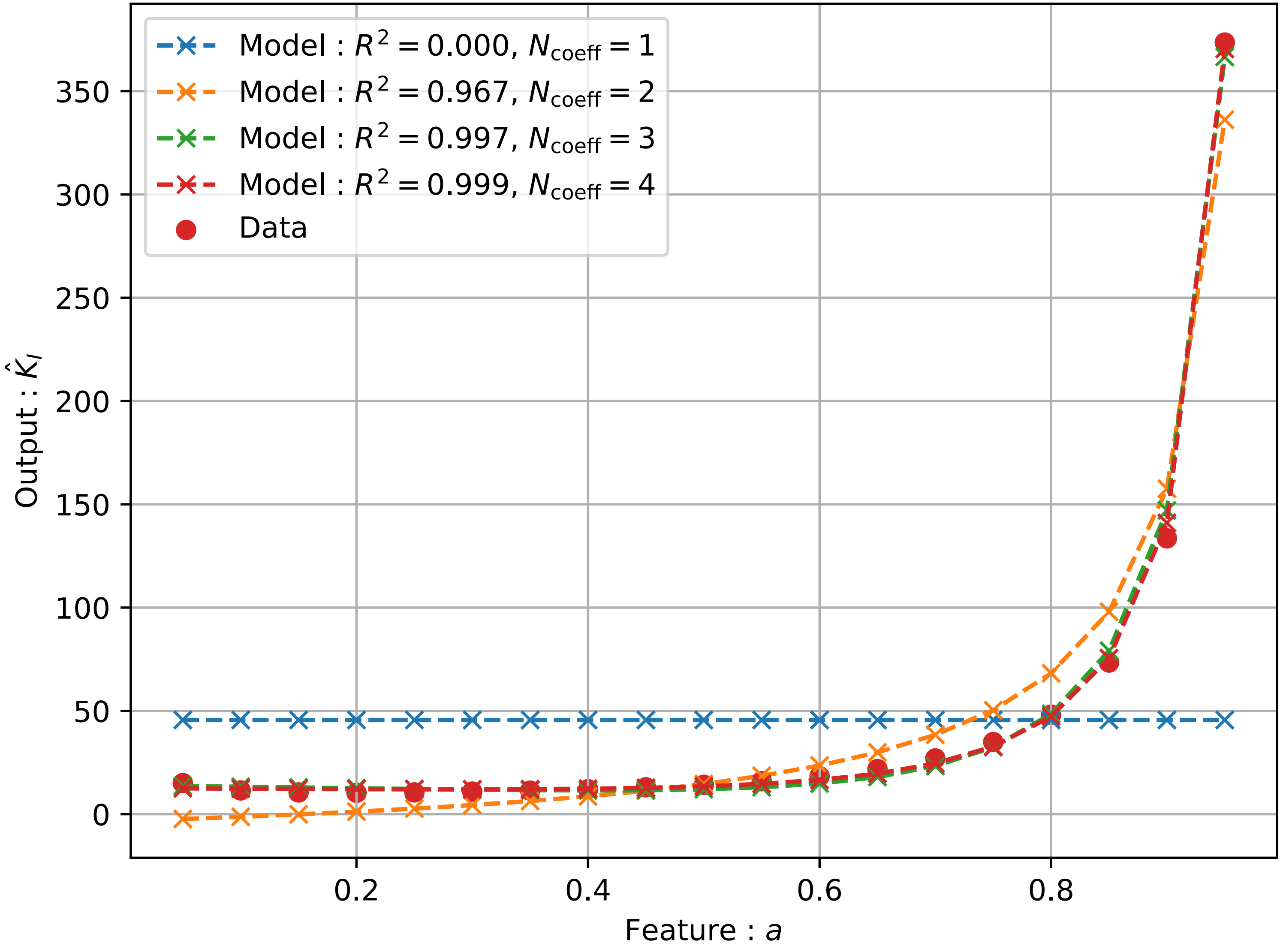
- Observations (elastic simulations) : $\left(\frac{a}{b}, \hat{K}_I\right)$

3. Sparse regression

$$\mathbf{p} = \left[\frac{a}{b}, \exp(a/b), \frac{1}{\sqrt{1-a/b}}\right]$$

- Hyperparameters :
  - Vary $\beta \to$ find good accuracy-simplicity tradeoff
  - $d = 2$, increase until satisfied.

Ex. 1 : CT specimen - Tradeoff - $d = 2$



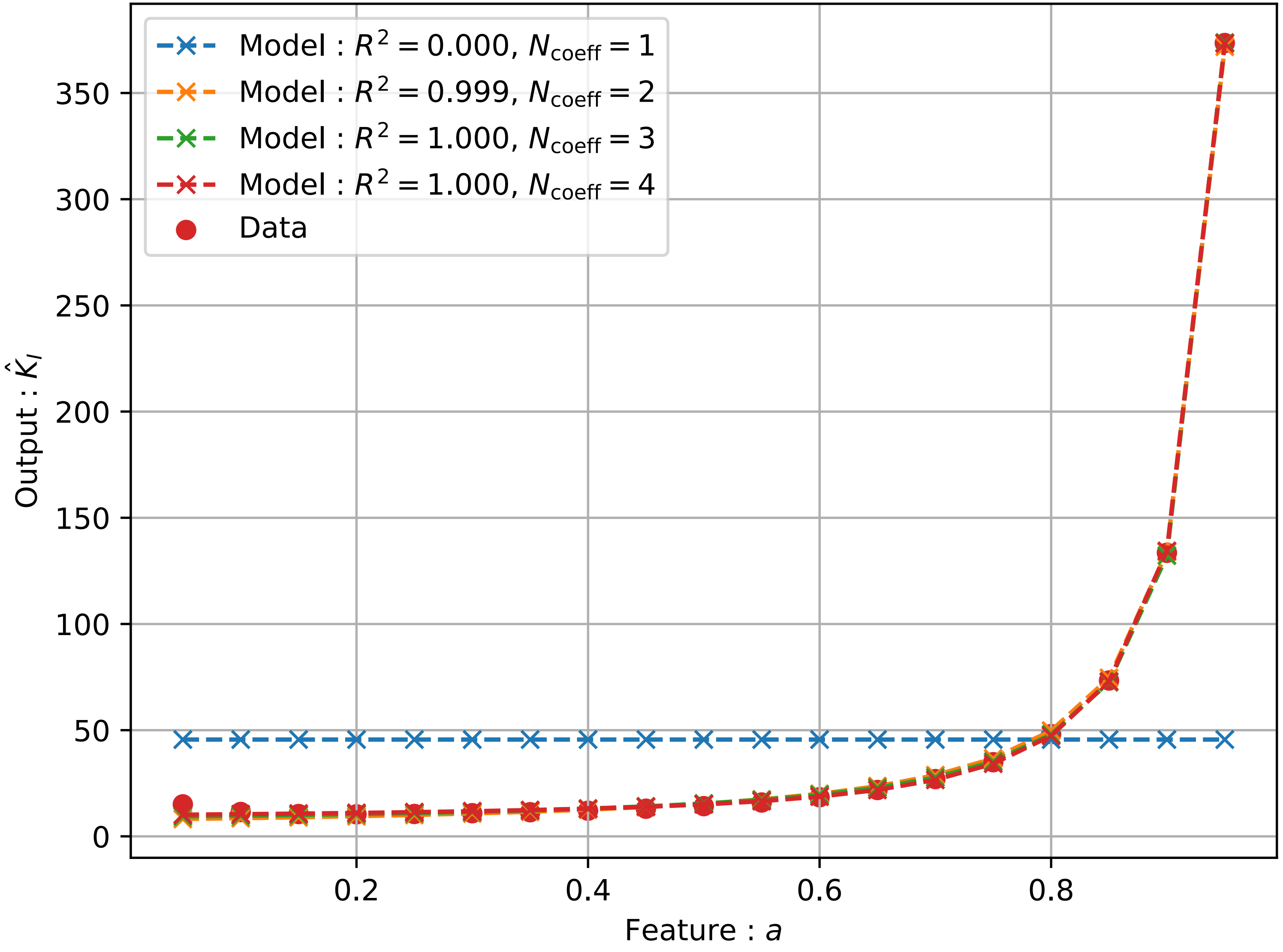
Code output

```
=== Model 1
Assessment :
    - Hyperparameter beta    : 3.83e+02
    - Number of coefficients : 1
    - R2 score               : 0.000
LaTeX expression:
\hat{K}_{I}(a/b) =
    +45.6

=== Model 2
Assessment :
    - Hyperparameter beta    : 1.02e+01
    - Number of coefficients : 2
    - R2 score               : 0.967
LaTeX expression:
\hat{K}_{I}(a/b) =
    -21.1
    +17.9 \frac{1}{\sqrt{1-a/b}}^2

=== Model 3
Assessment :
    - Hyperparameter beta    : 5.11e-01
    - Number of coefficients : 3
    - R2 score               : 0.997
LaTeX expression:
\hat{K}_{I}(a/b) =
    +5.43
    -14.4 \exp(a/b)^2
    +22.9 \frac{1}{\sqrt{1-a/b}}^2
```

Ex. 1 : CT specimen - Tradeoff - $d = 3$



Code output

```
=== Model 1
Assessment :
     - Hyperparameter beta     : 1.79e+03
     - Number of coefficients : 1
     - R2 score               : 0.000
LaTeX expression:
\hat{K}_{I}(a/b) =
     +45.6

=== Model 2
Assessment :
     - Hyperparameter beta     : 3.43e+00
     - Number of coefficients : 2
     - R2 score               : 0.999
LaTeX expression:
\hat{K}_{I}(a/b) =
     +3.75
     +4.11 \frac{1}{\sqrt{1-a/b}}^3

=== Model 3
Assessment :
     - Hyperparameter beta     : 6.13e-01
     - Number of coefficients : 3
     - R2 score               : 1.000
LaTeX expression:
\hat{K}_{I}(a/b) =
     +5.32
     -0.383 \exp(a/b)^3
     +4.19 \frac{1}{\sqrt{1-a/b}}^3
```

Ex. 2 : Rotating structure - Setup & Data

Experimental setup

By G. Eliyahu-Yakir and P. Reis (EPFL)

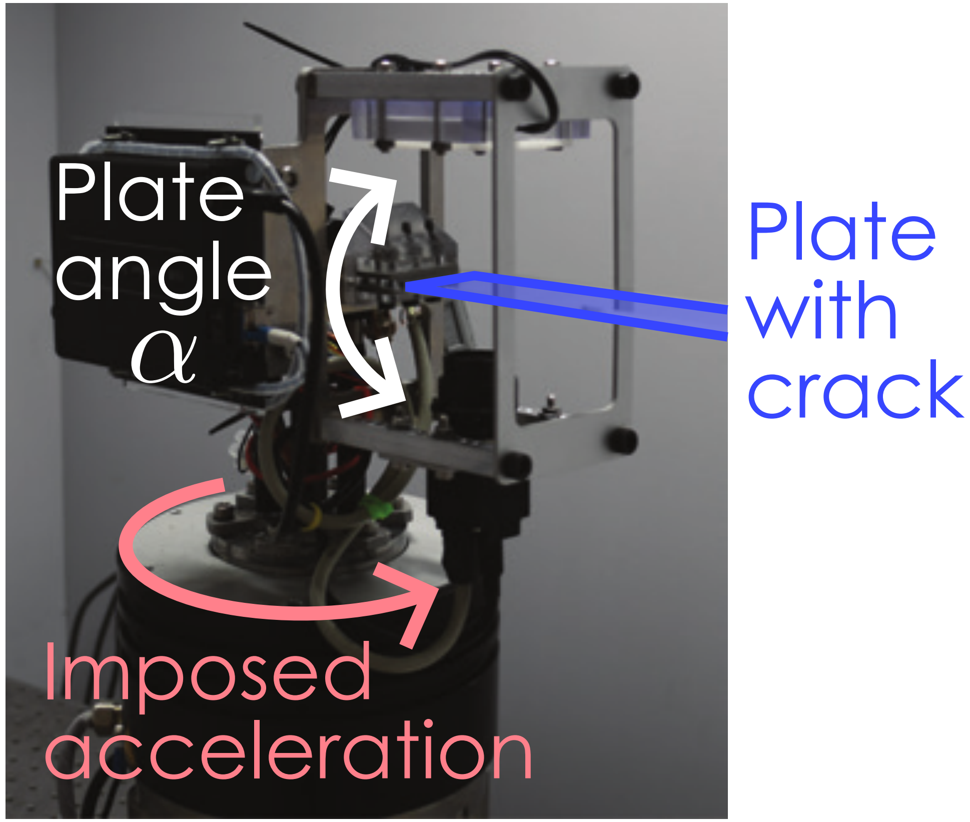
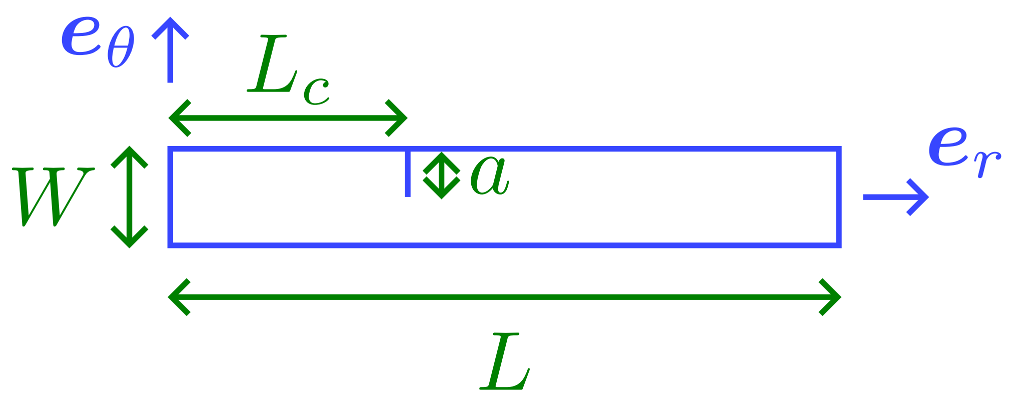
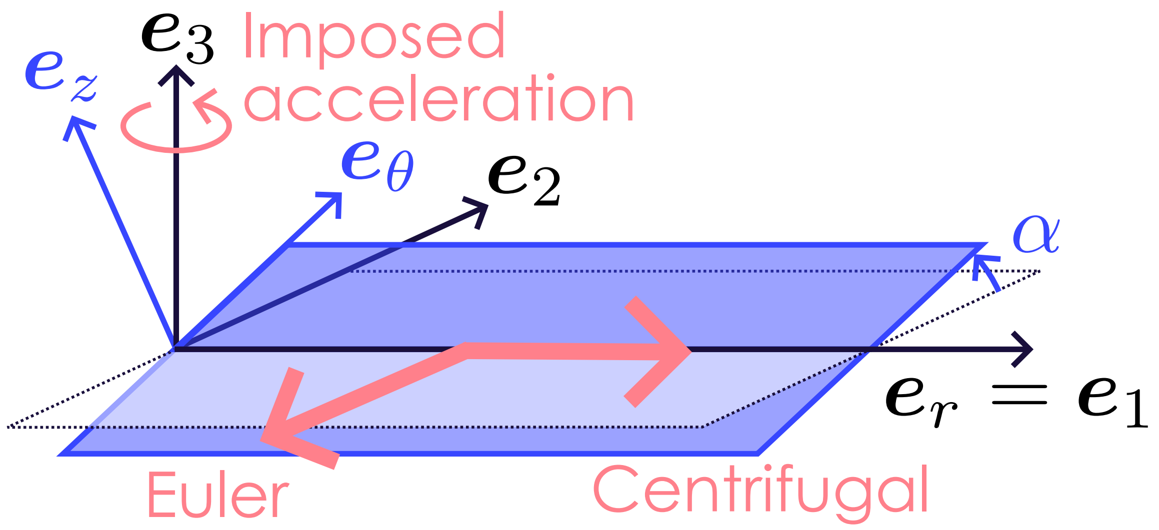


Plate geometry

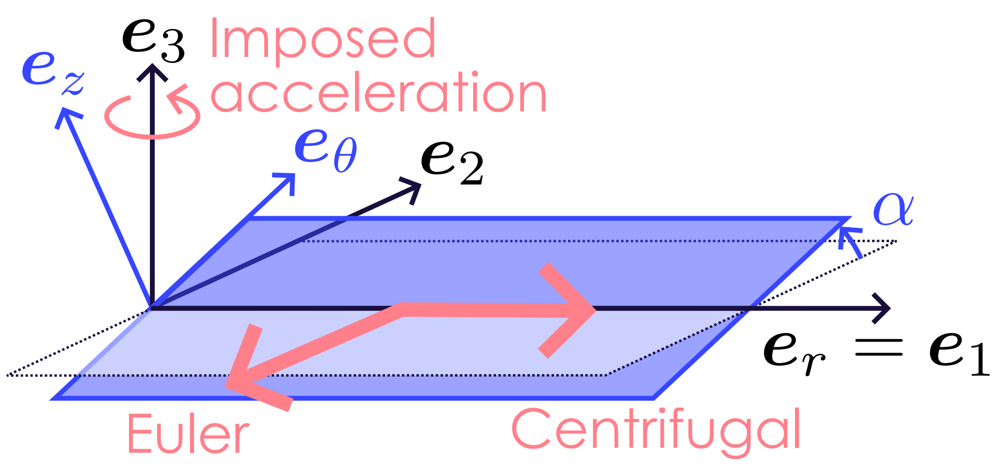


Illustration



Ex. 2 : Rotating structure - Physics

Illustration



7-parameter problem

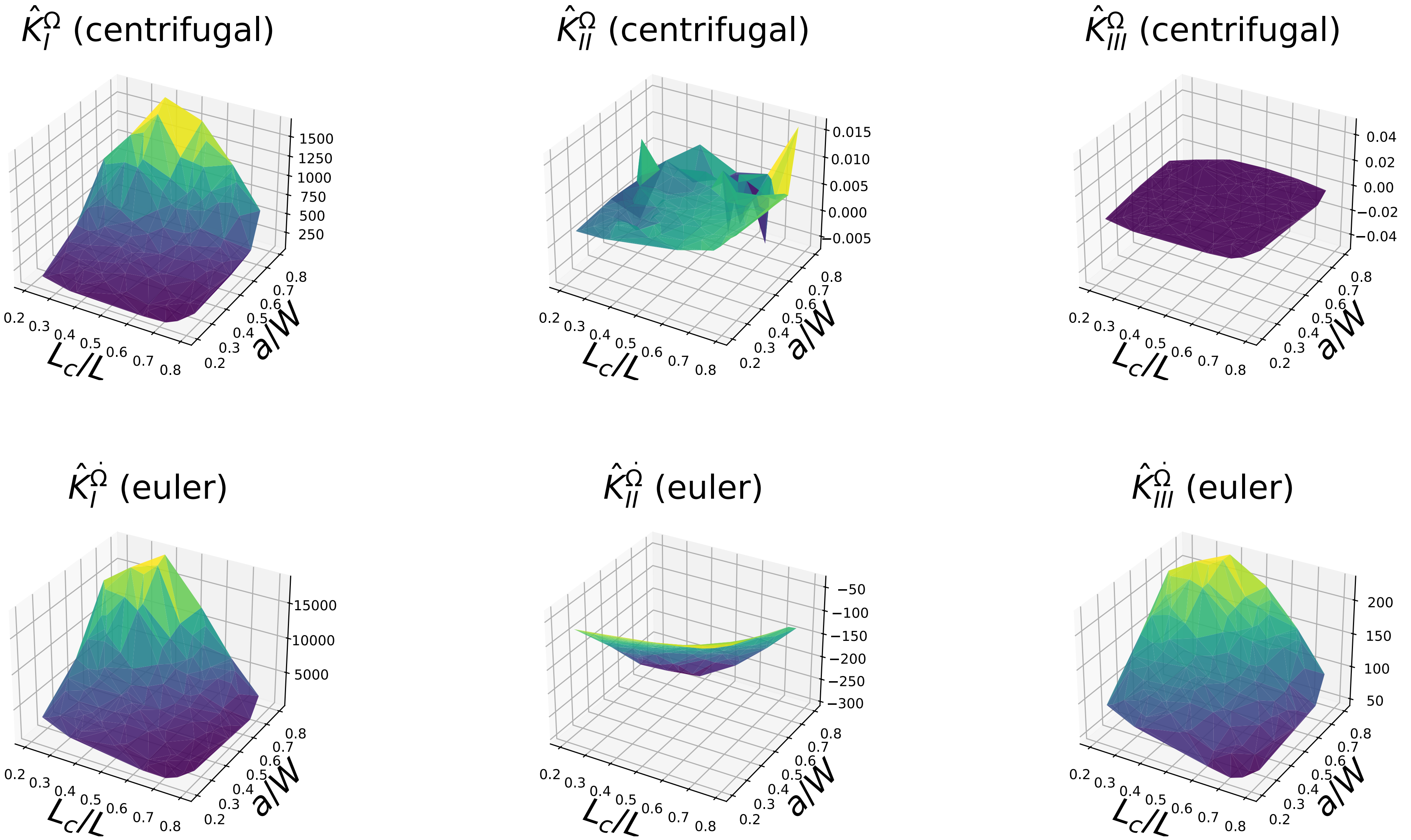
- Geometry: W, L, a, L_c.
- 3D volumic loads:
  - centrifugal : $\boldsymbol{f}_\Omega(r,\theta) \propto \Omega$
    - along $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$,
  - Euler : $\boldsymbol{f}_{\dot\Omega}(r,\theta) \propto \dot\Omega$
    - along $\boldsymbol{e}_2 = \cos(\alpha)\boldsymbol{e}_\theta - \sin(\alpha)\boldsymbol{e}_z$.

Dimensional analysis (towards a 3-parameter problem)

The superposition principle ($K_m = K_m^\Omega + K_m^{\dot\Omega}$) and the Buckingham Π theorem give ($\frac{L}{W}$ is fixed)

- centrifugal load
  - $K_I^\Omega = \rho\Omega^2 W^{5/2} \hat{K}_I^\Omega\left(\frac{a}{W}, \frac{L_c}{L}, \frac{L}{W}\right)$
  - $K_{II}^\Omega = \rho\Omega^2 W^{5/2} \hat{K}_{II}^\Omega\left(\frac{a}{W}, \frac{L_c}{L}, \frac{L}{W}\right)$
  - $K_{III}^\Omega = 0,$
- Euler load
  - $K_I^{\dot\Omega} = \rho\dot\Omega W^{5/2} \cos(\alpha)\hat{K}_I^{\dot\Omega}\left(\frac{a}{W}, \frac{L_c}{L}, \frac{L}{W}\right)$
  - $K_{II}^{\dot\Omega} = \rho\dot\Omega W^{5/2} \cos(\alpha)\hat{K}_{II}^{\dot\Omega}\left(\frac{a}{W}, \frac{L_c}{L}, \frac{L}{W}\right)$
  - $K_{III}^{\dot\Omega} = \rho\dot\Omega W^{5/2} \sin(\alpha)\hat{K}_{III}^{\dot\Omega}\left(\frac{a}{W}, \frac{L_c}{L}, \frac{L}{W}\right).$

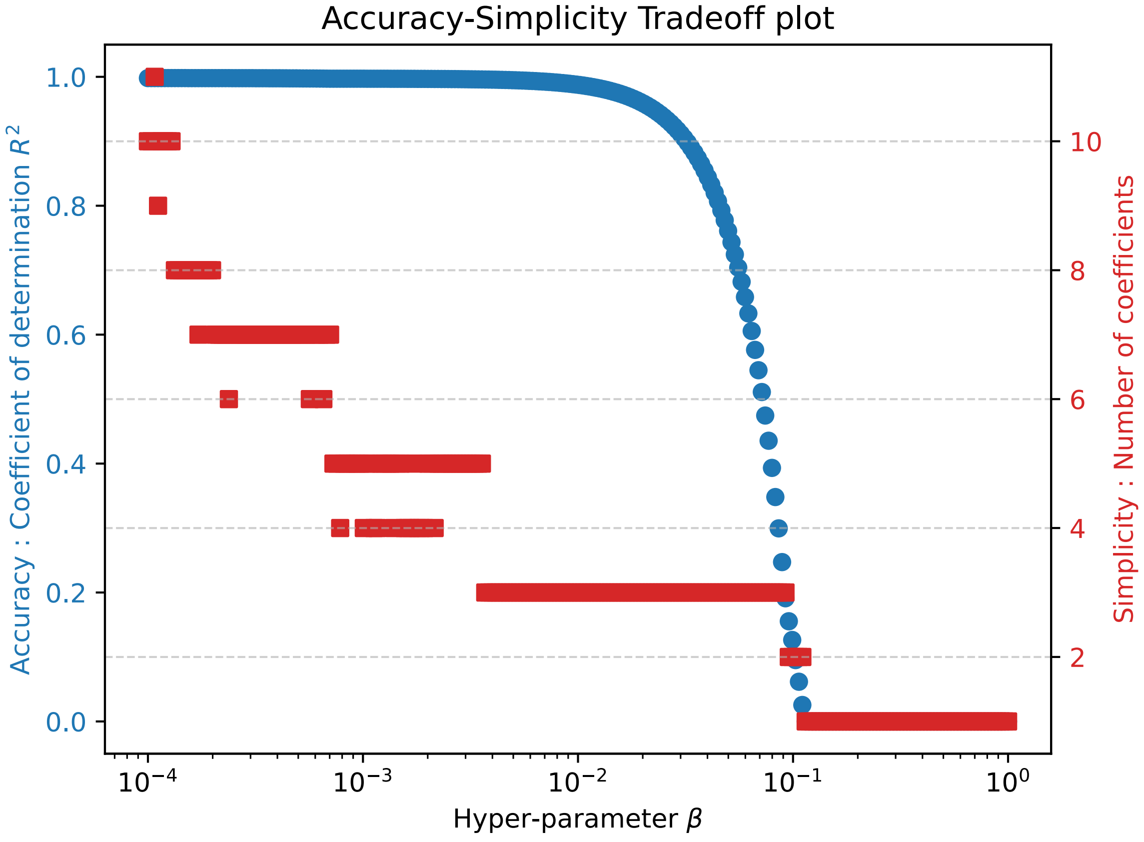
Ex. 2 : Rotating structure - Data

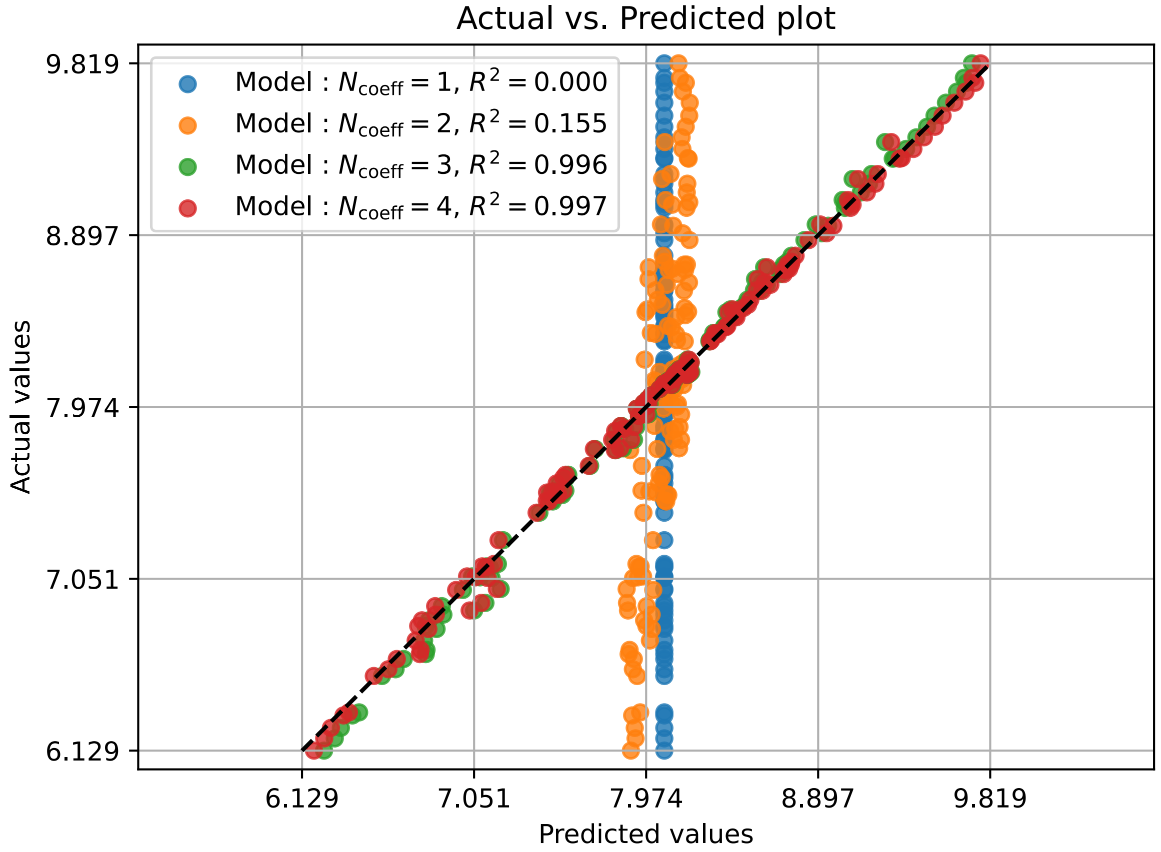


DATA GENERATION

Elastic FEM simulations

Ex. 2 : Rotating structure - Model $\hat{K}_I^{\dot{\Omega}}$





Ex. 2 : Rotating structure - All the SIFs

Model	N_{coeff}	R^2
$\hat{K}_I^{\Omega} = \exp\left(2.77 + 2.72\exp\left(\frac{a}{W}\right) - 0.986\exp\left(\frac{Lc}{L}\right)\right)$	3	0.995
$\hat{K}_{II}^{\Omega} = 0$	-	-
$\hat{K}_{III}^{\Omega} = 0$	-	
$\hat{K}_I^{\dot{\Omega}} = \exp\left(8.25 + 3.25\left(\frac{a}{W}\right)^2 - 3.9\left(\frac{L_c}{L}\right)^2\right)$	3	0.996
$\hat{K}_{II}^{\dot{\Omega}} = -5.75 - 127\left(\frac{a}{W}\right)^2 - 394\frac{a}{W}\left(1 - \frac{Lc}{L}\right)$	3	0.989
$\hat{K}_{III}^{\dot{\Omega}} = 52.7 + 238\left(\frac{a}{W}\right)\left(1 - \frac{Lc}{L}\right) + 97.4\left(\frac{a}{W}\right)^3 - 66.6\left(\frac{L_c}{L}\right)^3$	4	0.995

The fracture mechanics problem is now fully analytical.

Conclusion

Summary of the methodology

1. Physics : Dimensional analysis + Superposition
   - Isolate independent SIFs
   - Find the proper dimensionless parameters
2. Generation of the data
   - Sampling of the parametric space (LHS)
   - Elastic simulations
3. Machine learning : Sparse regression
   - Large polynomial + L1-regularization
   - Injection of non-linear terms as model input
4. Result : Sparse analytical approximation of the SIFs

Benefits in fracture mechanics

A problem of fracture can now be fully analytical facilitating

- uncertainty quantification,
- design (parametric optimization of structure),
- stability analysis,
- etc.

Thanks for your attention !

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Presentation

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Références

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